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Transformations of the boundary-layer equations for rotating flows

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The boundary layer on an axisymmetric surface above which the flow is rotating about the axis of symmetry is considered. Transformations of the governing equations which permit the generalizations of a known solution for one meridian shape in incompressible flow to a family of meridian shapes are shown to exist. For compressible flow, a transformation of the Stewartson–Illingworth type was found which reduces a compressible flow problem to an incompressible case. Also, remarks are made concerning the invariance of the turbulent boundary-layer integral equations assuming particular semi-empirical shear laws.

1. Formulation of the problem

Steady laminar axisymmetric boundary layers are considered here, with the outer flow, or the wall surface, or both rotating about the axis of symmetry. If the outer flow rotates, the circumferential velocity outside the boundary layer, V, is assumed to be independent of the axial co-ordinate z, but no *a priori* restriction is made on the radial distribution, i.e. $V = V(r)$. Since these steady inviscid flows are incompatible with any radial motions (except for $rV = \text{const.}$), any meridional motion is required to occur inside the boundary layer (even in the permissive case $rV = \text{const.}$, including $V = 0$).

The radius R of the wall surface may be given either as a function of z, or of the arclength s along the meridian. With boundary-layer co-ordinates s, n introduced in a meridional plane, the corresponding velocity components u, w can be obtained from a streamfunction $\psi(s,n)$:

$$\rho u R = \frac{\partial \psi}{\partial n}, \quad \rho w R = -\frac{\partial \psi}{\partial s}. \tag{1}$$

Let the boundary-layer equations for a compressible medium be given in the von Mises form:‡

$$\frac{u}{R^2}\frac{\partial u}{\partial s} = \left(\frac{\gamma^2}{\Gamma^2} - \frac{\rho_\infty}{\rho}\right)\frac{\Gamma^2}{R^5}\frac{dR}{ds} + u\frac{\partial}{\partial \psi}\left(\rho\mu u \frac{\partial u}{\partial \psi}\right), \tag{2}$$

$$\frac{1}{R^2}\frac{\partial \gamma}{\partial s} = \frac{\partial}{\partial \psi}\left(\rho\mu u\frac{\partial \gamma}{\partial \psi}\right), \tag{3}$$

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‡ These equations are not valid in the limit $dR/ds \to 0$, i.e. for nearly cylindrical shapes.

where $\gamma = vR$ is used instead of the circumferential velocity v inside the boundary layer. The boundary conditions are

$$u(0) = u(\infty) = 0 \tag{4}$$

in accordance with previously stated restrictions, and

$$\gamma(0) = \gamma_w, \quad \gamma(\infty) \equiv \Gamma = VR. \tag{5}$$

The energy equation is considered only for a perfect gas with Prandtl number 1, in which case it can be written in the form

$$\frac{\partial H}{\partial s} = R^2 \frac{\partial}{\partial \psi}\left(\rho\mu u \frac{\partial H}{\partial \psi}\right), \tag{6}$$

where

$$H = c_p T + \tfrac{1}{2}(u^2 + v^2) \tag{7}$$

is the stagnation or total enthalpy. Boundary conditions are

$$H(0) = H_w, \quad H(\infty) \equiv H_\infty = c_p T_\infty + \frac{\Gamma^2}{2R^2}. \tag{8}$$

The first aim of this paper is to present transformations of these boundary-layer equations which leave them invariant. Such co-ordinate transformations will be shown to exist; they permit the generalization of a known solution for one meridian shape to a set of solutions for a family of meridian shapes.

Next, a transformation of the Stewartson–Illingworth type (with attendant restrictions) will be given, which reduces a compressible boundary-layer problem to the incompressible case, albeit on a transformed meridian shape.

Finally, brief remarks will be made on the turbulent case, under the assumption that certain semi-empirical shear laws are valid.

2. Transformations which leave the boundary-layer equations unchanged

The dynamic similarity properties of the boundary-layer problem are well known; it is therefore no loss of generality to restrict attention to transformations involving the same medium, and to exclude the trivial transformation involving full geometric similarity. Non-trivial (but well known) is a transformation which may be called 'quasi-affine':

$$s' = \alpha s, \quad R' = \beta R, \tag{9}$$

with α and β constant. The prefix 'quasi' is used because the meridian shape $R(z)$ is not affinely transformed, except for the family of cones. Equations (2) and (3) are made independent of α and β by using the following variables:

$$u' = u, \quad \gamma' = \beta\gamma, \quad \Gamma' = \beta\Gamma, \quad \psi' = \beta\alpha^{\frac{1}{2}}\psi \tag{10}$$

and, in view of (1),

$$n' = \alpha^{\frac{1}{2}} n. \tag{11}$$

The best-known application of this transformation is the generalization of solutions obtained for a flat rotating wall to the family of cones (with possible singular effects at $R = 0$ ignored).

The invariance considerations can be extended to include compressibility and the energy equation. However, it is more convenient to explore the case of a constant property medium first, and to reconsider compressibility later.

To obtain a non-trivial transformation, a solution of (2) and (3) is assumed to be known; all variables pertaining to the known solution are characterized by the subscript '1'. It is seen from (2) and (3) that a condition for the reducibility of a general case (no subscripts) to the known case (subscript 1) is

$$R^2 ds = R_1^2 ds_1. \tag{12}$$

The quantity involved in this equation is recognized as the differential of the Mangler variable, which in non-rotating flow ($\gamma = \Gamma = 0$) reduces the problem to the plane case. However, the meridian shape does not enter first-order boundary-layer theory in the case of non-rotating flow; the problem under consideration is non-existent in this case.

For rotating flow, it is essential that the pressure gradient term in (2), which explicitly involves the meridian slope, remains unchanged. Inspection of the first right-hand side term in (2) gives the conditions, upon consideration of the relation already established in (12), that

$$\frac{dR}{R^3} = \frac{dR_1}{R_1^3}. \tag{13}$$

This can be integrated to give

$$\frac{1}{R^2} = \frac{1}{R_1^2} + A \tag{14}$$

and the constant of integration A is the free parameter of the transformation now established by (12) and (14). The differential relation between ds/dR and ds_1/dR_1 follows from (12) and (13):

$$\frac{ds}{dR} = \frac{ds_1}{dR_1}\left(\frac{R_1}{R}\right)^5 \tag{15}$$

and finally, with (14),

$$\frac{ds}{dR} = \frac{ds_1}{dR_1}(1 - AR^2)^{-\frac{5}{2}}. \tag{16}$$

For a given shape $R_1(s_1)$, the slope ds_1/dR_1 has to be expressed by R_1, and finally by R, using (14); a differential equation for a family of shapes is obtained, with A as the free parameter. $R_1(s_1)$ is recovered for $A = 0$.

A further generalization can be obtained by the combination of this transformation with the quasi-affine relations given by (9). It is seen from (15) or (16) that no loss of generality occurs upon applying the transformation to the variables s and R only, and by normalizing it with $\beta = 1$. The transformation thus generated can be obtained by replacing (12) by

$$R^2 ds = \alpha R_1^2 ds_1 \tag{12a}$$

while leaving (13) and (14) unchanged. Equation (16) becomes

$$\frac{ds}{dR} = \alpha \frac{ds_1}{dR_1}(1 - AR^2)^{-\frac{5}{2}}. \tag{16a}$$

The rules of this transformation are completed by putting

$$u = u_1, \quad \gamma = \gamma_1, \quad \Gamma = \Gamma_1, \quad \psi = \alpha^{\frac{1}{2}}\psi_1 \tag{17}$$

and from (1)

$$n = \alpha^{\frac{1}{2}}\frac{R_1}{R}n_1, \quad w = \alpha^{\frac{1}{2}}\frac{R_1}{R}w_1. \tag{18}$$

The definition of the generalized problem is completed by the calculation of $\Gamma(R)$, which together with $R(s)$ is a 'generating function' of the problem. As stated above, the relation is $\Gamma = \Gamma_1$, and by use of (14) this means

$$\Gamma = \Gamma_1\left(\frac{R}{(1-AR^2)^{\frac{1}{2}}}\right), \tag{19}$$

i.e., in the known function $\Gamma_1(R_1)$, R_1, given by (14), has to be inserted. Thus the generalization involves a redistribution of the rotation in the outer flow, except for the important case $\Gamma_1 = \text{const.}$

If the wall rotates, the boundary value γ_w has to be transformed in the same manner as Γ. Solutions of interest in this case involve almost exclusively rigid-body rotation, and this mode is not preserved by the transformation. Therefore no examples with moving walls will be considered.

Finally, the meridian shape in the z, R co-ordinate system can be obtained upon integrating the differential relation

$$\frac{dz}{dR} = \left[\alpha^2\left(\frac{ds_1}{dR_1}\right)^2\frac{1}{(1-AR^2)^5} - 1\right]^{\frac{1}{2}}. \tag{20}$$

Applications will be restricted to the case where the known solution is assumed on a flat surface, $ds_1/dR_1 = 1$. Equation (16a) can be integrated and gives

$$s = \alpha(1-\tfrac{2}{3}AR^2)(1-AR^2)^{-\frac{3}{2}}R. \tag{21}$$

A trivial constant is absorbed in the origin of s. For the conventional z, R representation of the meridian shape, which follows from (20), no closed integral was found. Numerical calculations were carried out for three cases with $A > 0$, and one case with $A < 0$.

If $A > 0$, it is most convenient to put $A = 1/R^{*2}$, and to scale the result with R^*; all meridians approach a cylinder with radius R^* asymptotically. For $\alpha = 1$, a 'cup-shaped' meridian results as shown in figure 1. If $\alpha < 1$, there is a hole in the middle with radius $R_h = R^*(1-\alpha^{\frac{2}{5}})^{\frac{1}{2}}$. The meridian has zero slope at $R = R_h$ and develops a cup-shape approaching $R = R^*$, as shown in figure 2 for $\alpha^2 = \frac{1}{2}$. In the case $\alpha > 1$, the cup is closed again, but has a cone-shaped bottom with the slope $\alpha^2 - 1$ for $R = 0$. Figure 3 gives the shape for $\alpha^2 = 2$.

For $A < 0$, solutions exist only for $\alpha > 1$, and within a circle with radius R_m (say), where $dz/dR = 0$. In order to scale the resulting meridian with R_m, define

$$A = -\frac{\alpha^{\frac{2}{5}} - 1}{R_m^2}. \tag{22}$$

Figure 4 shows the shape with $\alpha^2 = 2$; the centre is conical, as in figure 3, but the meridian ends abruptly with a horizontal slope at $R = R_m$.

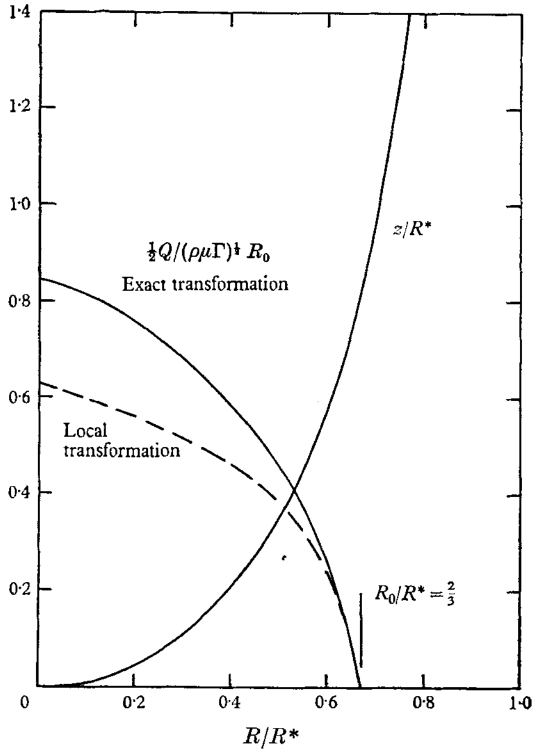

FIGURE 1. Meridian shape z for $A > 0$, $\alpha^2 = 1$, and the radial mass flux Q for a boundary layer beginning at $R_0 = \frac{2}{3}R^*$ with constant circulation in the outer flow.

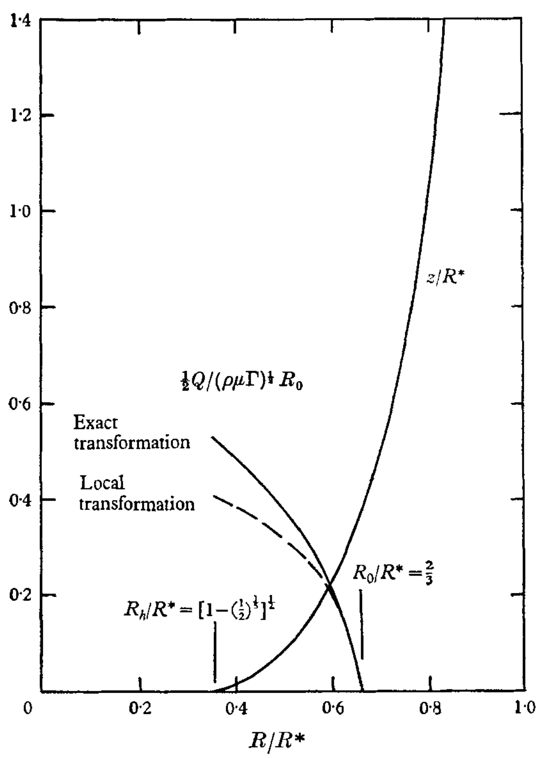

FIGURE 2. Meridian shape z for $A > 0$, $\alpha^2 = \frac{1}{2}$, and the radial mass flux Q for a boundary layer beginning at $R_0 = \frac{2}{3}R^*$ with constant circulation in the outer flow.

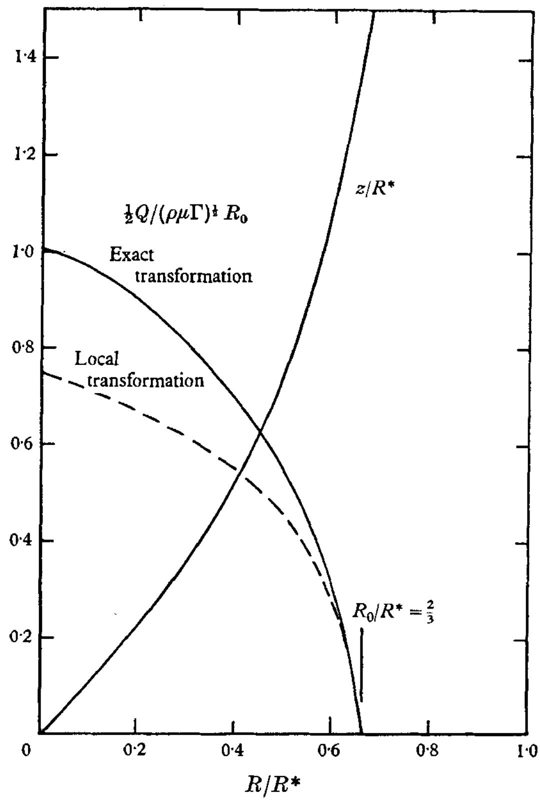

FIGURE 3. Meridian shape z for $A > 0$, $\alpha^2 = 2$, and the radial mass flux Q for a boundary layer beginning at $R_0 = \frac{2}{3}R^*$ with constant circulation in the outer flow.

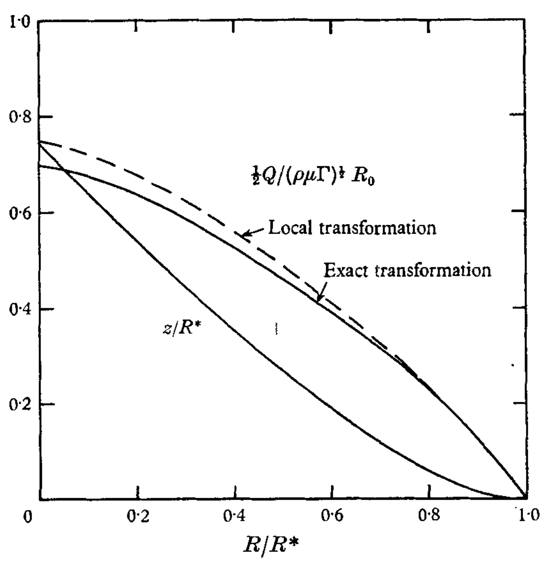

FIGURE 4. Meridian shape z for $A < 0$, $\alpha^2 = 2$, and the radial mass flux Q for a boundary layer beginning at $R_m = R^*$ with constant circulation in the outer flow.

The remaining steps which lead to explicit results will be demonstrated for a specific known solution over a flat surface, namely for a potential vortex flow over a finite flat disk. This problem, which was first solved by a momentum integral method by Taylor (1950), has since been treated more accurately by several authors; a series expansion method applied by Mack (1962) and a numerical solution by Anderson (1966) are particularly noteworthy. These solutions permitted a critical re-evaluation of momentum methods, and a simplified momentum integral technique (originally proposed by Anderson and Mack) has been found to give closed-form results of high accuracy. Critical discussion of the momentum integral method was given by Rott & Lewellen (1966), where further references to this subject are given.

A particularly important boundary-layer variable is $\psi(\infty) = Q$, which represents the total radial flux (divided by 2π) in the boundary layer. The explicit momentum integral result for this quantity is, for arbitrary meridian shape and Γ distribution, with the wall at rest and the boundary-layer origin at R_0,

$$Q = 1{\cdot}56(\rho\mu)^{\frac{1}{2}}\Gamma^{\lambda_1-1}\left\{\int_{R_0}^{R} \Gamma^{2-\frac{4}{3}\lambda_1}\left(\frac{ds}{dR}\right)^{\frac{2}{3}} R^{\frac{1}{3}}\,dR\right\}^{\frac{3}{4}}, \tag{23}$$

where $\lambda_1 = 4{\cdot}93$ is a profile constant. It is interesting to note that this formula does have the invariance property with respect to the proposed transformation, as

$$(ds^2 R\,dR)^{\frac{1}{3}} = (\alpha^2 ds_1^2 R_1\,dR_1)^{\frac{1}{3}} \tag{24}$$

is easily verified, and, with $\Gamma = \Gamma_1$, the formula gives for Q (which has to transform like ψ) $Q = \alpha^{\frac{1}{2}}Q_1$. This result is to be expected from the derivation of (23). The formula can be applied approximately to any meridian shape; in this sense, the 'exact' transformation is not needed for its use. However, suppose that (23) has been tested by comparison with a known exact solution and has been found sufficiently accurate. Then the cases generalized by the proposed transformation are just as accurate, and the use of (23) gives simple closed-form results.

The known solution is taken to be the case of a vortex of constant strength Γ_1, over a disk of the radius R_{01}, for which (23) yields the 'tested' result

$$Q_1 = 1{\cdot}26(\rho\mu\Gamma_1)^{\frac{1}{2}}R_{01}\left[1-\left(\frac{R_1}{R_{01}}\right)^{\frac{4}{3}}\right]^{\frac{3}{4}}. \tag{25}$$

To generalize this result for the meridian shapes presented before, a radius R_0 has to be chosen which will be the outer edge for the generalized flow. With $R_0/R^* = \eta$, the relation between R_0 and R_{01} is (according to (14), for $A > 0$)

$$R_{01} = \frac{R_0}{(1-\eta^2)^{\frac{1}{2}}} \tag{26}$$

and the general correspondence between R and R_1 is given by

$$\frac{R_1}{R_{01}} = \frac{R}{R_0}\left(\frac{1-\eta^2}{1-\eta^2R^2/R_0^2}\right)^{\frac{1}{2}}. \tag{27}$$

$Q = \alpha^{\frac{1}{2}}Q_1$ follows from (25) with $\Gamma = \Gamma_1$ and the insertion of R_{01} and R_1 by use of (26) and (27). The normalized flux $Q/R_0(\rho\mu\Gamma_1)^{\frac{1}{2}}$ is plotted in figures 1 to 3 for

$\eta = \frac{2}{3}$. These results can be considered 'exact', in the sense of the preceding discussion.

If $A < 0$, the general relation between R and R_1 is renormalized in the form

$$R_1 = \frac{R}{\left(1 + \frac{\alpha^{\frac{2}{5}} - 1}{R_m^2} R^2\right)^{\frac{1}{2}}}. \tag{28}$$

Let this example be restricted to the case when $R_0 = R_m$, so that, from (28),

$$R_{01} = R_m \alpha^{-\frac{1}{5}}. \tag{29}$$

The normalized flux found from $Q = \alpha^{\frac{1}{2}} Q_1$ is plotted in figure 4.

3. Comparison with results of an approximate transformation

A different interesting 'local' approximation can be obtained as follows: if the left-hand side terms of (2) and (3) are changed by inserting $\partial/\partial s = (\partial/\partial R)\,(dR/ds)$, it is possible to eliminate dR/ds approximately by putting

$$\psi \left(\frac{dR}{ds}\right)^{\frac{1}{2}} = \overline{\psi}, \tag{30}$$

while the exact application of the variable $\overline{\psi}$ naturally gives rise to additional terms in (2) and (3), except for $dR/ds = \text{const.}$ Approximately, however, the explicit appearance of s is avoided by these operations, so that, in terms of the unchanged variable R,

$$Q = \left(\frac{ds}{dR}\right)^{\frac{1}{2}} \overline{Q}(R). \tag{31}$$

Now $\overline{Q}$ is the known solution for a flat surface, taken as the standard reference case. The interest in this approximation stems from the fact that the flux variation in Ekman layers with dR/ds has exactly the form given by (31), and can be calculated 'locally'.

The approximate formula (31) applied to the examples treated previously gives results shown in figures 1–4. Initially, the predictions of the local and the exact transformations are the same. However, for the three cases corresponding to $A > 0$, a significant departure occurs as the centre is approached, with the approximate formula underestimating the mass flux. The agreement is seen to be good over the entire surface for the one case with $A < 0$.

4. Transformations which reduce the compressible flow equations to the incompressible case

In this section, the equations are reconsidered with variable properties. A necessary condition for the reducibility of (2) to the constant property case is $\rho\mu = \rho_\infty \mu_\infty = \text{const.}$ across the boundary layer, or $\mu \sim T$; this assumption is made herein. It is seen from (2) that, if $\Gamma = 0$, then this condition is sufficient to reduce the general problem to the constant property case even if $\gamma \neq 0$, i.e. for rotating surfaces with the outer flow at rest; the Howarth–Dorodnitsyn transformation is valid. If $\Gamma \neq 0$, the coupling of (2) to the energy equation occurs through the appearance of ρ_∞/ρ in the pressure gradient term. In principle, this also happens in non-rotating flow, and the method known from the work of Illingworth (1949) and Stewartson (1949) will be used to eliminate this coupling.

Again, the transformation is only successful for a gas with Prandtl number 1, for which (6) holds; it has a solution

$$H = A\gamma + B, \tag{32}$$

with A and B constant, as comparison with (3) shows. If $A \neq 0$, there is heat transfer to the wall; these solutions have been noted and explored by Ohrenberger (1967), Barcilon (1966) and Anderson (1966). Here, however, only the case $A = 0$ are considered; this restriction completes the necessary conditions for the desired transformation. For the flow, this means that it is isoenergic; the enthalpy outside the boundary layer is

$$c_p T_\infty \equiv h_\infty = H - \frac{\Gamma^2}{2R^2} \tag{33}$$

and with the relation

$$\frac{1}{\rho_\infty}\frac{dp}{dR} = \frac{\Gamma^2}{R^3} \tag{34}$$

the state of the outer flow is determined, as soon as $\Gamma(R)$ is specified. As is well known, an isentropic relation is found for $\Gamma = \text{const.}$; however, the validity of the proposed transformation is not restricted to this special case. For example, if

$$\Gamma = cR^m \tag{35}$$

and the gas is perfect (so that $(\kappa - 1)h = p/\rho$), it follows from (33) and (34) that

$$\frac{p}{p_s} = \left(\frac{h_\infty}{H}\right)^{\{\kappa/(\kappa-1)\}\{1/(1-m)\}}, \tag{36}$$

where p_s is the stagnation pressure and H is the stagnation enthalpy. Actually, to find the means of realization for a non-isentropic but isoenergic outer flow is an open question. Examples will be restricted therefore to the irrotational iso-energic–isentropic case, which can be realized by the superposition of a small sink to the outer rotating flow. Its strength has to be small enough so that $u(\infty) \ll V$, and $u(\infty) = 0$ is still applicable approximately as outer boundary condition.

Inside the boundary layer, the explicit solution for the temperature is

$$c_p T \equiv h = H - \frac{\gamma^2}{2R^2} - \frac{u^2}{2}$$
$$= h_\infty + \frac{\Gamma^2 - \gamma^2}{2R^2} - \frac{u^2}{2}. \tag{37}$$

It is seen that the wall is adiabatic and at the constant temperature H/c_p if it is non-rotating; this is, however, not the only case for which the proposed transformation is applicable. Only the validity of (37) is a necessary condition.

Important for the transformation of the pressure term in (2) is the relation

$$\frac{\gamma^2}{\Gamma^2} - \frac{\rho_\infty}{\rho} = \frac{\gamma^2}{\Gamma^2} - \frac{h}{h_\infty}$$
$$= \left(\frac{\gamma^2}{\Gamma^2} - 1\right)\frac{H}{h_\infty} - \frac{u^2}{2h_\infty}. \tag{38}$$

The transformation to the incompressible case is now achieved in two steps. First, the velocity is transformed by the relation

$$u = F(s)\,u_i, \tag{39}$$

where now the subscript i refers to the (known) incompressible case. On the left-hand side of (2), a term quadratic in u_i occurs, namely

$$\frac{1}{R^2} F \frac{dF}{ds} u_i^2 \tag{40}$$

which has to compensate a right-hand side term in u_i^2. The latter appears when the bracket in the pressure gradient term is replaced by the expression given by (38), and has the form

$$F^2 \frac{u_i^2}{2h_\infty} \frac{\Gamma^2}{R^5} \frac{dR}{ds} = \frac{F^2}{R^2} \frac{u_i^2}{2h_\infty} \frac{1}{\rho_\infty} \frac{dp}{ds}, \tag{41}$$

where the second expression results by the use of (34). The terms (40) and (41) cancel if

$$\frac{dF}{F} = \frac{dp}{2\rho_\infty h_\infty} = \frac{\kappa - 1}{2\kappa} \frac{dp}{p}. \tag{42}$$

for a perfect gas, or

$$F = \left(\frac{p}{p_s}\right)^{(\kappa-1)/2\kappa} = \left(\frac{h_\infty}{H}\right)^{1/\{2(1-m)\}} \tag{43}$$

with the reference pressure chosen as p_s. The second expression is only valid for power-law velocity distributions, and follows from (36).

Thus far, (2) and (3) have been changed to

$$\frac{u_i}{R^2} \frac{\partial u_i}{\partial s} = \left(\frac{\gamma^2}{\Gamma^2} - 1\right) \frac{H\Gamma^2}{h_\infty F^2 R^5} \frac{dR}{ds} + \rho_\infty \mu_\infty u_i F \frac{\partial}{\partial \psi}\left(u_i \frac{\partial u_i}{\partial \psi}\right), \tag{44}$$

$$\frac{1}{R^2} \frac{\partial \gamma}{\partial s} = \rho_\infty \mu_\infty F \frac{\partial}{\partial \psi}\left(u_i \frac{\partial u_i}{\partial \psi}\right). \tag{45}$$

The second step, which reduces these equations to the incompressible form (subscript i) for a constant-property medium at the stagnation-point state of the compressible medium (subscript s), is completed by the use of the transformation

$$\gamma = \gamma_i, \quad \Gamma = \Gamma_i, \quad \psi = \psi_i, \tag{46}$$

$$\rho_\infty \mu_\infty F R^2 ds = \rho_s \mu_s R_i^2 ds_i, \tag{47}$$

$$\frac{H}{h_\infty} \frac{1}{F^2} \frac{dR}{R^3} = \frac{dR_i}{R_i^3}. \tag{48}$$

Equation (47) gives what may be considered as a generalization of the Mangler variable, while (48) reduces the pressure gradient term to the incompressible form. The latter relation has a remarkably simple integral, namely

$$R_i = F(R)\,R, \tag{49}$$

which can be verified by use of (34). Not contained in this result is a possible generalization which would include the transformation of the incompressible or the compressible case onto itself, as discussed in the previous section. It is proposed to use a 'normalized' transformation to the incompressible case as given by (49), and to admit the 'compound' use of this relation with (14) (as well as with the quasi-affine relation) to obtain the most general transformation. Equations (47) and (48) lead to the relation

$$\frac{ds_i}{dR_i} = \frac{ds}{dR}\frac{h_\infty \rho_\infty \mu_\infty}{H \rho_s \mu_s}\frac{1}{F^2} = \frac{ds}{dR}\frac{\mu_\infty}{\mu_s}\left(\frac{p}{p_s}\right)^{1/\kappa}. \tag{50}$$

It is possible to discuss the use of (50) for arbitrary viscosity–temperature laws; this necessitates the use of an additional 'reference-temperature' approximation, in order to obtain (44) and (45). However, applications are restricted here to case $\mu_\infty/\mu_s = h_\infty/H$, for which the transformation, as defined by (39), (43), (46), (49) and (50), is exact.

To find the meridian shape in the incompressible case, the right-hand side of (50) has to be expressed by R_i. From (32), (43) and (49), it follows for the power-law Γ-distributions given by (35) that

$$\frac{h_\infty}{H} = \frac{R_i^{2(1-m)}}{(c/2H)+R_i^{2(1-m)}} \tag{51}$$

and (50) becomes

$$\frac{ds_i}{dR_i} = \frac{ds}{dR}\left(\frac{h_\infty}{H}\right)^{1+\{1/(\kappa-1)\}\{1/(1-m)\}}$$

$$= \frac{ds}{dR}\left(\frac{p}{p_s}\right)^{1-m(\kappa-1)/\kappa}. \tag{52}$$

It is seen that ds_i/dR_i is less than ds/dR for all m is less than $\kappa/(\kappa-1) \geqslant \frac{5}{2}$. This occurs both for the isentropic case $m = 0$ and for rigid rotation $m = 2$.

Thus, for the most interesting cases of compressible flow over a flat surface ($ds/dR = 1$), no 'real' meridian shape can be constructed for the incompressible case with the normalization agreed upon thus far, since $ds_i/dR_i < 1$.

It is now possible to invoke the generalizations of a given solution to different meridian shapes, and to change from an unrealistic meridian to a realizable case. However, the subsequent discussion will be restricted to the momentum integral solution of the problem and (23) is considered as the solution of the incompressible case. This formula has been shown to be invariant with respect to the transformation discussed in §2, so that its effect can be disregarded; when compounded with the transformation proposed in this section, the latter will be the only one to have an effect on the form of (23).

The problem, then, is to think of (23) rewritten with subscript i and to find the generalization to the compressible case by use of the transformation established in this section.

The result is

$$Q = 1{\cdot}56(\rho_s\mu_s)^{\frac{1}{2}}\Gamma^{\lambda_1-1}\left\{\int_R^{R_0}\Gamma^{2-\frac{4}{3}\lambda_1}\left(\frac{\rho_\infty\mu_\infty}{\rho_s\mu_s}\right)^{\frac{2}{3}}\left(\frac{H}{h_\infty}\right)^{\frac{1}{3}}\left(\frac{ds}{dR}\right)^{\frac{2}{3}}R^{\frac{1}{3}}\,dR\right\}^{\frac{3}{4}}. \tag{53}$$

To prove this formula, it is best to employ the original set of transformation equations, namely (46), (47) and (48). Such a derivation shows that, for the proof of (53) only, the function F and the integrated equation (49) need not be known.

As a first application, consider the initial behaviour of Q in the vicinity of R_0, with R near enough to R_0 so that the variation of the integrand in (53) can be disregarded. If Q_i denotes the flux calculated with constant property stagnation state values, then near R_0

$$Q = \left(\frac{\rho_\infty \mu_\infty}{\rho_s \mu_s}\right)^{\frac{1}{2}} \left(\frac{H}{h_\infty}\right)^{\frac{1}{4}} Q_i. \tag{54}$$

The first factor merely changes the reference state from the stagnation point to the state at R_0, while the second factor represents a 'genuine' effect of the local Mach number, based on the swirling velocity.

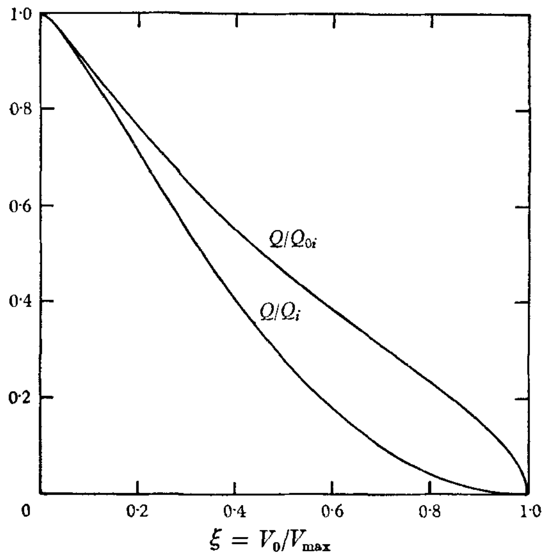

FIGURE 5. Normalized radial mass flux to the hollow core of a compressible potential vortex on a flat surface of finite radius in air.

For the second example, consider Γ = constant over a flat surface. The outer flow is isentropic, and for a perfect gas, with $\mu \sim T$, equation (53) becomes (by use of (33))

$$Q = 1{\cdot}56(\rho_s \mu_s \Gamma)^{\frac{1}{2}} \left[\int_R^{R_0} \left(1 - \frac{\Gamma^2}{2HR^2}\right)^{\frac{1}{3}(\kappa+1)/(\kappa-1)} R^{\frac{1}{3}} dR\right]^{\frac{3}{4}}. \tag{55}$$

The radius of the hollow core of the compressible potential vortex is $R_c = \Gamma(2H)^{-\frac{1}{2}}$. The total flux at R_c is of interest; it depends on the ratio $\xi = R_c/R_0 = V_0/V_{max}$. For $\kappa = 1{\cdot}4$ (air), the exponent of the first factor inside the integral becomes 2, and the integration is elementary. Results are shown in figure 5; compared with Q_i (obtained for $\xi = 0$), Q shows a steady decline. Also included in figure 5 is a comparison with Q_{0i} which is obtained when freestream properties at the initial radius R_0 are used in a constant property formula, thus,

$$\frac{Q_{0i}}{Q_i} = \left(\frac{\rho_{\infty 0} \mu_{\infty 0}}{\rho_s \mu_s}\right)^{\frac{1}{2}} = \left(\frac{p_0}{p_s}\right)^{\frac{1}{2}} = (1 - \xi^2)^{\kappa/\{2(\kappa-1)\}}. \tag{56}$$

The ratio Q/Q_{0i} is plotted and shows the variation of the radial mass flux with ξ when the static conditions at R_0 are held fixed.

The extension of the calculation to R_c is naturally an idealization. For instance, it should be kept in mind that the small sink, which makes the isentropic case physically realistic, changes R_c to a 'limiting line' at a somewhat different radius. Moreover, near this limiting line viscous effects will become important in the outer flow even in those cases when they are negligible elsewhere. However, these effects do not change the flow except very near to R_c.

All these results are valid only if (i) $\rho\mu$ = constant, (ii) the Prandtl number is 1, and (iii) the wall is adiabatic if at rest; (37) is valid for moving walls. When the first two restrictions are violated, the resulting errors can be estimated, and corrections can be worked out. The last restriction is of a more fundamental nature. Nevertheless, the proposed transformation is useful for simplifying the equations in the case of compressible boundary layers with heat transfer; several applications were given by Ohrenberger (1967).

5. Remark on turbulent flow

In retrospect, the method used in this paper can be summarized as follows: transformations were derived from the boundary-layer equations for laminar flow, but applications were restricted to the use of momentum integral methods. It is clear that the proper transformation of the momentum integral equations is a necessary but not a sufficient condition for the invariance of the full equations. It can be shown, however, that the invariance consideration of the constant-property tangential and radial momentum integral equations does lead to the same transformations, (12) and (14), as that of the full equations, but that the approximate solution, (23), is invariant with respect to a much more general family of transformations.

These results make it interesting to investigate the invariance of the tangential and radial momentum integral equations with generalized shear terms, which can cover laminar as well as certain semi-empirical turbulent shear laws (Rott & Lewellen 1966). The outcome of this investigation, which can be easily carried out on the basis of the equations given by Rott & Lewellen, is the following: (13) and its integrated form, (14), remain unchanged, but (12) has to be modified for different shear laws; the general formula is

$$R^{2\epsilon} ds = R_1^{2\epsilon} ds_1, \tag{57}$$

where the exponent ϵ represents the power law dependence of the proposed shear law on the Reynolds number based on the boundary-layer thickness. For instance, $\epsilon = 1$ for laminar flow, while $\epsilon = \frac{1}{4}$ is appropriate for a Blasius-type turbulent shear law, which was first adapted to rotating flows (with smooth surfaces) by von Kármán (1921).

The quasi-affine transformation is also affected; in (10) and (11) $\alpha^{\frac{1}{2}}$ has to be replaced by the factor $\alpha^{1/(1+\epsilon)}$. The generalization of (23) has been given by Rott & Lewellen.

For plane compressible high-speed turbulent boundary layers, the successful semi-empirical notion of a 'turbulent Prandtl number equal to 1' has led to a series of highly speculative theories, which generalize transformation laws obtained in compressible laminar flow to the turbulent case; a wealth of experimental results can be summoned in support of such theoretical results. The same speculations could be applied in the rotating flow case, but, in the absence of experimental high-speed turbulent boundary-layer results, their application cannot be justified, and no further discussion of this generalization is offered.

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